ELSEVIER Journal of Geometry and Physics 39 (2001) 265–275

JOURNAL OF GEOMETRY AND PHYSICS

Metric-affine gravity and the Nester–Witten 2-form

Marco Godina [a], Paolo Matteucci [b,*], James A. Vickers [b]

[a] *Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy*
[b] *Faculty of Mathematical Studies, University of Southampton, Highfield, Southampton SO17 1BJ, UK*

Received 4 September 2000; received in revised form 18 December 2000

Abstract

In this paper we redefine the well-known metric-affine Hilbert Lagrangian in terms of a spin connection and a spin-tetrad. On applying the Poincaré–Cartan method and using the geometry of gauge-natural bundles, a global gravitational superpotential is derived. On specializing to the case of the Kosmann lift, we recover the result originally found by Kijowski [Gen. Rel. Gravity 9 (1978) (10) 857] for the metric (natural) Hilbert Lagrangian. On choosing a different, suitable lift, we can also recover the Nester–Witten 2-form, which plays an important role in the energy positivity proof and in many quasi-local definitions of mass. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 83C40; 83D05

Subj. Class.: General relativity

Keywords: Hilbert–Einstein Lagrangian; Spinors; Gauge-natural; Superpotential

1. Introduction

Conserved quantities have always represented an intriguing issue in general relativity, as was pointed out by Penrose [21] in a very famous paper. The jet bundle formalism provides an adequate framework for Lagrangian field theories and the Poincaré–Cartan method enables one to associate with each of them globally conserved charges (cf., e.g. [8,18,25]). In particular, for first-order theories these charges are uniquely defined, and in the second-order case, although uniqueness is lost, still there is a unique *canonical* choice.

Natural Lagrangian field theories have been known for a long time, Einstein's general relativity being one of them. Many physical theories, though, such as Yang–Mills and Dirac theories, are *non-natural*, i.e. the "configuration bundle", which is nothing but the space of the dependent variables or "fields", is not a natural bundle. Roughly speaking, *natural*

* Corresponding author.
E-mail address: p.matteucci@maths.soton.ac.uk (P. Matteucci).

0393-0440/01/$ – see front matter © 2001 Elsevier Science B.V. All rights reserved.
PII: S0393-0440(01)00007-9

bundles (such as the tangent or the cotangent bundle) form a particular class of fibre bundles, where, once a coordinate change on the base manifold is given, the corresponding fibred coordinate change is known. More technically, natural bundles can be regarded as fibre bundles associated to higher order frame bundles on manifolds (cf. [16]).

If we aim at considering the coupling of a natural theory, such as general relativity, with a non-natural one, we are *sometimes* forced to "redefine" our field variables in order to make the coupling physically meaningful. In particular, if we want to describe the interaction and feedback between gravity and spinor fields, *spin-tetrads*, and not tetrads, are the appropriate objects to be considered (cf. [3,9]). *Gauge-natural bundles* provide a suitable geometrical framework for such objects. Such bundles are fibre bundles associated to "abstract" principal bundles with arbitrary structure group (cf. [16]).

The superpotential associated with the standard Hilbert Lagrangian for general relativity, or the "Hilbert superpotential", was first given by Kijowski [12] using ideas developed by Kijowski himself [11] and Kijowski and Szczyrba [13,14]. It was also derived explicitly using the Poincaré–Cartan method by Kijowski and Tulczyjew [15], in a Hamiltonian (multisymplectic) framework, and Ferraris et al. [7], in the Lagrangian context. [1] In [5] the authors were able to reformulate the previous result of two of them in terms of tetrads. But, again, their theory was still natural, and this meant there was no real advantage of such a reformulation.

Recently, two parallel papers [3,9] addressed the problem of re-expressing the above results in terms of spin-tetrads and coupling *true* general relativity with Fermionic matter, but their findings implicitly relied on a Poincaré–Cartan form associated with a particular ("quasi-natural") lift of vector fields onto the bundle of orthonormal frames, the "Kosmann lift" (cf. [2]).

In this paper we redefine the *metric-affine* Hilbert Lagrangian in terms of a spin connection and a spin-tetrad. The ensuing superpotential is genuinely "general", in the sense that it is derived in a *completely* gauge-natural context and also allows for the presence of torsion.

Such a reformulation enables us not only to single out the aforementioned link between the Hilbert superpotential and the Kosmann lift, but also to associate the well-known Nester–Witten 2-form with another particular lift, thereby providing us with a clear-cut geometric interpretation of a rather famous but somewhat obscure integrand in general relativity. This lift turns out to be essentially the *dual* of the Kosmann lift. For a different characterization of the Nester–Witten 2-form see, e.g., the detailed analysis by Dubois-Violette and Madore [1].

The structure of the paper is as follows: in Section 1 we recall the main ingredients of the Poincaré–Cartan method, in Section 2 we set up the geometric framework of our theory, and in Section 3 we derive our main results.

Finally, in Section 4 we present a first-order covariant Lagrangian for general relativity and derive the relevant superpotential.

[1] There exists an extensive literature on both the multisymplectic formulation of field theories and its Lagrangian counterpart. The interested reader is referred to [10], on pp. 15 and 25–26, respectively.

2. Poincaré–Cartan method

It is well-known that to each *first-order* Lagrangian there corresponds a *unique* global Poincaré–Cartan form. Let M be an (orientable, Hausdorff, paracompact, smooth) m-dimensional manifold and

$$\mathcal{L} : J^1 B \to \bigwedge^m T^* M, \qquad \mathcal{L} : j^1 y^{\mathfrak{a}} \mapsto \mathcal{L}(j^1 y^{\mathfrak{a}}) \equiv L(x^{\alpha}, y^{\mathfrak{a}}, y^{\mathfrak{a}}{}_{\lambda})\, \mathrm{d}s$$

a first-order Lagrangian defined on the first-order jet prolongation $J^1 B$ of a gauge-natural bundle B over M (cf. [16, Section 51]), $y^{\mathfrak{a}}$ being a section of B and $\mathrm{d}s \equiv \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{m-1}$ the standard volume form on M. Define its *momenta* as

$$f_{\mathfrak{a}}{}^{\mu} := \frac{\partial L}{\partial y^{\mathfrak{a}}{}_{\mu}}.$$

The *Poincaré–Cartan form* associated to $\mathcal{L}$ is then given by

$$\Theta(\mathcal{L}) := \mathcal{L} + f_{\mathfrak{a}}{}^{\mu}\, \mathrm{d}_{\mathrm{V}} y^{\mathfrak{a}} \wedge \mathrm{d}s_{\mu},$$

where d_{V} is the vertical differential (notably, $\mathrm{d}_{\mathrm{V}} y^{\mathfrak{a}} = \mathrm{d}y^{\mathfrak{a}} - y^{\mathfrak{a}}{}_{\mu}\, \mathrm{d}x^{\mu}$: cf. [8]) and we set $\mathrm{d}s_{\mu} := \partial_{\mu} \rfloor \mathrm{d}s$, '$\rfloor$' denoting the inner product.

The knowledge of the Poincaré–Cartan form enables us to calculate the so-called *Noether current* of the Lagrangian in question. Indeed, if one has a one-parameter subgroup of automorphisms of B generated by a projectable vector field Ξ (with projection ξ onto M), the Noether current associated to $\mathcal{L}$ along the vector field Ξ is given by

$$E(\mathcal{L}, \Xi) := -\mathrm{Hor}[J^1 \Xi \rfloor \Theta(\mathcal{L})] = -\xi \rfloor \mathcal{L} + f_{\mathfrak{a}}{}^{\mu} \pounds_{\Xi} y^{\mathfrak{a}}\, \mathrm{d}s_{\mu},$$

where Hor denotes the horizontal projection (cf. [8, Section 3.1]), $J^1 \Xi$ is the first-order jet prolongation of Ξ, and the well-known relation

$$J^1 \Xi \rfloor \mathrm{d}_{\mathrm{V}} y^{\mathfrak{a}} = -\pounds_{\Xi} y^{\mathfrak{a}}$$

between vertical differential and (generalized) Lie derivative is used in obtaining the second equation (cf. [16, Section 47]).

3. Geometric framework

Let M be an orientable, Hausdorff, paracompact, smooth, four-dimensional manifold. Suppose M admits Lorentzian metrics of signature -2, i.e. assume that M satisfies the topological requirements which ensure the existence on it of Lorentzian structures [$\mathrm{SO}(1,3)^e$-reductions]. Let $\mathbb{L}(M)$ be the (principal) bundle of linear frames over M with structure group $\mathrm{GL}(4, \mathbb{R})$.

Assume now that M admits a *free spin structure* $(\Sigma, \tilde{\Lambda})$, i.e. the existence of at least one principal fibre bundle Σ over M with structure group $\mathrm{Spin}(1,3)^e \cong \mathrm{SL}(2, \mathbb{C})$, called the *spin structure bundle*, and at least one strong (i.e. covering the identity map) equivariant morphism $\tilde{\Lambda} : \Sigma \to \mathbb{L}(M)$ [9]. We call the bundle map $\tilde{\Lambda}$ a *spin-frame* on Σ.

This definition of a spin structure induces metrics on M. Indeed, given a spin-frame $\tilde{\Lambda} : \Sigma \to \mathbb{L}(M)$, we can define a metric via the reduced subbundle $\mathrm{SO}(M, g) \equiv \tilde{\Lambda}(\Sigma)$ of $\mathbb{L}(M)$. In other words, the *dynamic* metric $g \equiv g_{\tilde{\Lambda}}$ is defined to be the metric such that frames in $\tilde{\Lambda}(\Sigma) \subset \mathbb{L}(M)$ are g-orthonormal frames. It is important to stress that in our picture the metric g is built up a posteriori, after a spin-frame has been determined by the field equations in a way which is compatible with the (free) spin structure one has used to define spinors.

Now let Λ be the epimorphism which exhibits $\mathrm{Spin}(1,3)^e$ as a twofold covering of $\mathrm{SO}(1,3)^e$ and consider the following left action of the group $\mathrm{GL}(4,\mathbb{R}) \times \mathrm{Spin}(1,3)^e$ on the manifold $\mathrm{GL}(4,\mathbb{R})$

$$\begin{aligned}
&\rho : (\mathrm{GL}(4,\mathbb{R}) \times \mathrm{Spin}(1,3)^e) \times \mathrm{GL}(4,\mathbb{R}) \to \mathrm{GL}(4,\mathbb{R}),\\
&\rho : ((A^{\mu}{}_{\nu}, S^{a}{}_{b}), u^{a}{}_{\mu}) \mapsto u'^{a}{}_{\mu} := (\Lambda(S))^{a}{}_{b} u^{b}{}_{\nu} (A^{-1})^{\nu}{}_{\mu}
\end{aligned}$$

together with the associated bundle $\Sigma_\rho := W^{1,0}(\Sigma) \times_\rho \mathrm{GL}(4,\mathbb{R})$, where $W^{1,0}(\Sigma) := \mathbb{L}(M) \times_M \Sigma$ denotes the principal prolongation of order $(1,0)$ of the principal fibre bundle Σ (cf. [16, Section 52.4]). The bundle $W^{1,0}(\Sigma)$ is a principal fibre bundle with structure group $\mathrm{GL}(4,\mathbb{R}) \times \mathrm{Spin}(1,3)^e$. It turns out that Σ_ρ is a fibre bundle associated to $W^{1,0}(\Sigma)$, i.e. a gauge-natural bundle of order $(1,0)$. A section of Σ_ρ will be called a *spin-tetrad*.

Recall now that a *principal connection* on a principal fibre bundle $P(M, G)$ may be regarded as a G-equivariant global section of the affine jet bundle $J^1P \to P$, where the G-action on J^1P is induced by the first jet prolongation of the canonical (right) action of G on P (cf. [8, Section 2.7]). Owing to G-equivariance there is a 1–1 correspondence between principal connections and global sections of the quotient bundle $J^1P/G \to M$.

More specifically, let $P = \Sigma$ and let $\mathfrak{spin}(1,3) \cong \mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ denote the Lie algebra of $\mathrm{Spin}(1,3)^e$. Consider then the following left action on the vector space $V_C := \mathfrak{spin}(1,3) \otimes (\mathbb{R}^4)^*$

$$\begin{aligned}
&\lambda : (\mathrm{GL}(4,\mathbb{R}) \times T^1_4 \mathrm{Spin}(1,3)^e) \times V_C \to V_C,\\
&\lambda : ((A^{\mu}{}_{\nu}, S^{a}{}_{b}, S^{a}{}_{b\mu}), u^{a}{}_{b\mu}) \mapsto u'^{a}{}_{b\mu}\\
&\quad := (A^{-1})^{\nu}{}_{\mu} [(\Lambda(S))^{a}{}_{c} u^{c}{}_{d\nu} (\Lambda(S^{-1}))^{d}{}_{b} - (\Lambda(S))^{a}{}_{c\nu} (\Lambda(S^{-1}))^{c}{}_{b}],
\end{aligned}$$

where $(\Lambda(S))^{a}{}_{c\nu}$ are the components of $j^1_0(\Lambda \circ S)$, i.e. an element of $T^1_4\mathrm{SO}(1,3)^e$, and $S : \mathbb{R}^4 \to \mathrm{Spin}(1,3)^e$ is a local map defined around the origin $0 \in \mathbb{R}^4$. Hence define the associated bundle $C := W^{1,1}(\Sigma) \times_\lambda V_C$, where $W^{1,1}(\Sigma) := \mathbb{L}(M) \times_M J^1\Sigma$ denotes the principal prolongation of order $(1,1)$ of Σ (cf. [16, Section 52.4]). It turns out that C is a gauge-natural bundle of order $(1,1)$ isomorphic to $J^1\Sigma/\mathrm{Spin}(1,3)^e \to M$. A section of C will be called a *spin connection*.

4. Metric-affine gravity

Let $\theta^a{}_\mu$ be a spin-tetrad and $\omega^a{}_{b\mu}$ a spin connection, as defined in the previous section. Set locally

$$\theta^a := \theta^a{}_\mu \,\mathrm{d}x^\mu, \qquad e_a := e_a{}^\mu \partial_\mu,$$

where $e_a{}^\mu$ is implicitly defined via the relation $\theta^a{}_\mu e_b{}^\mu = \delta^a{}_b$, and

$$\omega^a{}_b := \omega^a{}_{b\mu}\,\mathrm{d}x^\mu, \qquad \Omega^a{}_b := \mathrm{d_H}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b,$$

$\mathrm{d_H}$ being the horizontal differential (cf. [8, Section 3.1]); $\omega^a{}_b$ and $\Omega^a{}_b$ are recognized to be the (horizontal) connection 1-form and curvature 2-form, respectively.

We can now "redefine" the (metric-affine) *Hilbert Lagrangian* as

$$\begin{aligned}
&\mathcal{L} : \Sigma_\rho \underset{M}{\times} J^1 C \to \bigwedge^4 T^* M, \\
&\mathcal{L} : (\theta^a{}_\mu, j^1\omega^a{}_{b\mu}) \mapsto \mathcal{L}(\theta^a{}_\mu, j^1\omega^a{}_{b\mu}) := -\frac{1}{2\kappa}\Omega_{ab} \wedge \Sigma^{ab},
\end{aligned} \tag{4.1}$$

where $\kappa := 8\pi G/c^4$ and $\Sigma^{ab} := {}^*(\theta^a \wedge \theta^b)$. The equations of motion are obtained by varying $\mathcal{L}$ with respect to θ^c and ω_{ab}:

$$\frac{\delta\mathcal{L}}{\delta\theta^c} \equiv \frac{1}{2\kappa}\Omega_{ab} \wedge \Sigma^{ab}{}_c \equiv -\frac{1}{\kappa} G^a{}_c \Sigma_a = 0, \tag{4.2a}$$

$$\frac{\delta\mathcal{L}}{\delta\omega_{ab}} \equiv \frac{1}{2\kappa}\nabla\Sigma^{ab} = 0, \tag{4.2b}$$

where $\Sigma^{ab}{}_c := e_c \rfloor \Sigma^{ab}$, $\Sigma_a := \frac{1}{6} e_{abcd}\theta^b \wedge \theta^c \wedge \theta^d$ and ∇ denotes the (gauge-) covariant exterior derivative. We stress that the condition $\nabla\Sigma^{ab} = 0$ is equivalent to $T^a \equiv \nabla\theta^a = 0$, T^a being the torsion 2-form.

According to the definition given in Section 1, the appropriate Poincaré–Cartan form for Lagrangian (3.1) is

$$\Theta(\mathcal{L}) \equiv \mathcal{L} + \mathrm{d_V}\omega_{ab} \wedge \frac{\partial\mathcal{L}}{\partial\,\mathrm{d_H}\omega_{ab}} = \mathcal{L} - \frac{1}{2\kappa}\,\mathrm{d_V}\omega_{ab} \wedge \Sigma^{ab}, \tag{4.3}$$

where $\partial\mathcal{L}/\partial\,\mathrm{d_H}\omega_{ab}$ stands for $\partial L/\partial\omega_{ab\nu,\mu}\,\mathrm{d}s_{\mu\nu}$ and $\mathrm{d}s_{\mu\nu} := \partial_\nu \rfloor \mathrm{d}s_\mu$. Hence, the Noether current associated with a projectable vector field Ξ is

$$\begin{aligned}
E(\mathcal{L}, \Xi) &= -\xi\rfloor\mathcal{L} - \frac{1}{2\kappa}\pounds_\Xi\omega_{ab} \wedge \Sigma^{ab} \\
&\equiv \frac{1}{2\kappa}[(\xi\rfloor\Omega_{ab}) \wedge \Sigma^{ab} + \Omega_{ab} \wedge (\xi\rfloor\Sigma^{ab}) - \pounds_\Xi\omega_{ab} \wedge \Sigma^{ab}] \\
&\equiv \frac{1}{2\kappa}[(\xi\rfloor\Omega_{ab}) \wedge \Sigma^{ab} + \xi^c\Omega_{ab} \wedge \Sigma^{ab}{}_c - \pounds_\Xi\omega_{ab} \wedge \Sigma^{ab}].
\end{aligned} \tag{4.4}$$

Now, our configuration bundle B is $\Sigma_\rho \times_M C$, which is a gauge-natural bundle. Therefore, every (principal) automorphism $\Phi \in \mathrm{Aut}(\Sigma)$ induces an automorphism Φ_B on B. This holds also infinitesimally, i.e. for invariant (projectable) vector fields defined on Σ. Strictly speaking, an invariant vector field $\Xi \in \mathfrak{X}(\Sigma)$ defines functorially a projectable vector field $\Xi_B \in \mathfrak{X}(\Sigma_\rho \times_M C)$. Moreover, every $\mathrm{Spin}(1,3)^e$-invariant vector field $\Xi \in \mathfrak{X}(\Sigma)$ projects onto an $\mathrm{SO}(1,3)^e$-invariant vector field, which we denote by the same symbol $\Xi \in \mathfrak{X}(\Sigma/\mathbb{Z}_2)$. Since the natural projection $\mathrm{pr} : \Sigma \to \Sigma/\mathbb{Z}_2$ is a covering map (locally, a diffeomorphism) of principal fibre bundles, it follows that there is a bijection between projectable $\mathrm{SO}(1,3)^e$-invariant vector fields on $\Sigma/\mathbb{Z}_2$ and projectable $\mathrm{Spin}(1,3)^e$-invariant vector fields on Σ (cf. [2]). If a spin-frame is given, such a bijection extends to an invariant vector field bijection between $\Sigma/\mathbb{Z}_2$ and $\mathrm{SO}(M,g) \equiv \tilde{\Lambda}(\Sigma)$, and hence, between $\mathrm{SO}(M,g)$ and Σ. Yet, only the Lie derivative of the connection 1-form is needed here, so we can simply regard Ξ_B as belonging to $\mathfrak{X}(C)$. Then, a projectable vector field $\Xi_C \in \mathfrak{X}(C)$ onto a vector field $\xi \equiv \xi^\mu \partial_\mu \in \mathfrak{X}(M)$ reads as

$$\Xi_C = \xi^\mu \partial_\mu + \Xi^a{}_{b\mu} \frac{\partial}{\partial u^a{}_{b\mu}},$$

where

$$\Xi^a{}_{b\mu} := -(\partial_\mu \xi^\nu u^a{}_{b\nu} + u^a{}_{c\mu} \Xi^c{}_b - u^c{}_{b\mu} \Xi^a{}_c + \partial_\mu \Xi^a{}_b),$$

$\Xi \equiv \xi^\mu(x)\partial_\mu + \Xi^a{}_b(x)\alpha_a{}^b$ being the corresponding projectable vector field on $\Sigma/\mathbb{Z}_2$ and $(u^a{}_{b\mu})$ local fibre coordinates on C. The vector fields $\alpha_a{}^b$ are local right $\mathrm{SO}(1,3)^e$-invariant vector fields on $\Sigma/\mathbb{Z}_2$, which in a suitable chart $(x^\mu, u_a{}^b)$ read as

$$\alpha_a{}^b \equiv \tfrac{1}{2}(\rho_a{}^b - \eta^{bc}\eta_{ad}\rho_c{}^d),$$

η denoting the Minkowski metric and $\rho_a{}^b := u_c{}^b \partial/\partial u_c{}^a$. Therefore, the Lie derivative of $u^a{}_{b\mu} = \omega^a{}_{b\mu}(x)$ is just

$$\pounds_\Xi \omega^a{}_{b\mu} = \xi^\nu \partial_\nu \omega^a{}_{b\mu} + \partial_\mu \xi^\nu \omega^a{}_{b\nu} + \omega^a{}_{c\mu} \Xi^c{}_b - \omega^c{}_{b\mu} \Xi^a{}_c + \partial_\mu \Xi^a{}_b,$$

which can be readily recast in Cartan formalism as

$$\pounds_\Xi \omega^a{}_b = \xi \rfloor \Omega^a{}_b + \nabla \check{\Xi}^a{}_b, \tag{4.5}$$

$\check{\Xi}^a{}_b := \Xi^a{}_b + \omega^a{}_{b\mu}\xi^\mu$ being the vertical part of Ξ. On substituting (4.5) into (4.4), we finally get

$$\begin{aligned} E(\mathcal{L}, \Xi) &= \frac{1}{2\kappa}(\xi^c \Omega_{ab} \wedge \Sigma^{ab}{}_c - \nabla \check{\Xi}_{ab} \wedge \Sigma^{ab}) \\ &= \frac{1}{2\kappa}[\xi^c \Omega_{ab} \wedge \Sigma^{ab}{}_c + \check{\Xi}_{ab} \nabla \Sigma^{ab} - \mathrm{d_H}(\check{\Xi}_{ab} \Sigma^{ab})]. \end{aligned} \tag{4.6}$$

Now, by virtue of equations of motion (4.2a) and (4.2b),

$$U(\mathcal{L}, \Xi) := -\frac{1}{2\kappa} \check{\Xi}_{ab} \Sigma^{ab} \tag{4.7}$$

is recognized to be the *superpotential* associated with Lagrangian (4.1). This superpotential, which was derived in a *completely* gauge-natural context and — to the best of our knowledge — appears here for the first time, represents the most general superpotential possible in this metric-affine formulation of gravity (modulo, of course, closed 2-forms).

Note that in the case of the Kosmann lift [2] we have

$$\check{\Xi}_{ab} = (\check{\xi}_{\mathrm{K}})_{ab} \equiv -\nabla_{[a}\xi_{b]}, \tag{4.8}$$

which, substituted in (4.7), gives

$$U(\mathcal{L}, \xi_{\mathrm{K}}) = \frac{1}{2\kappa}\nabla_a\xi_b\Sigma^{ab}, \tag{4.9}$$

i.e. *half* of the well-known Komar potential [17], in accordance with the result found by Ferraris et al. [5] in a purely natural context. This is also the lift implicitly used by Godina et al. [9] in the two-spinor formalism.

Let now $\sigma_a^{AA'}$ denote the Infeld–van der Waerden symbols, which express the isomorphism between $\mathrm{Re}[S(M) \otimes \bar{S}(M)]$ and TM in the orthonormal basis induced by the spin-frame chosen (cf. [22,26]), and consider the following lift:

$$\xi^\mu = e_a{}^\mu\sigma^a{}_{AA'}\lambda^A\bar{\lambda}^{A'}, \quad \check{\Xi}_{ab} = (\check{\xi}_{\mathrm{W}})_{ab} := -4\sigma_{[a}{}^{AA'}\sigma_{b]}{}^{BB'}\,\mathrm{Re}(\bar{\lambda}_{B'}\nabla_{BA'}\lambda_A), \tag{4.10}$$

which will be referred to as the *Witten lift*. Then

$$U(\mathcal{L}, \xi_{\mathrm{W}}) = \mathrm{Re}W \equiv -\frac{2}{\kappa}\mathrm{Re}(\mathrm{i}\bar{\lambda}_{A'}\nabla\lambda_A \wedge \theta^{AA'}), \tag{4.11}$$

which is the (real) Nester–Witten 2-form [20,23]. Indeed, we have: [2]

$$\begin{aligned}\check{\Xi}_{ab}\Sigma^{ab} &= -2\bar{\lambda}_{B'}\nabla_{BA'}\lambda_A\Sigma^{ab} + \mathrm{CC} = 2\mathrm{i}^*(\bar{\lambda}_{A'}\nabla_{BB'}\lambda_A)\Sigma^{ab} + \mathrm{CC}\\ &= 2\mathrm{i}\bar{\lambda}_{A'}\nabla_b\lambda_A{}^*\Sigma^{ab} + \mathrm{CC} = -2\mathrm{i}\bar{\lambda}_{A'}\nabla_b\lambda_A\theta^a \wedge \theta^b + \mathrm{CC}\\ &= 2\mathrm{i}\bar{\lambda}_{A'}\nabla\lambda_A \wedge \theta^{AA'} + \mathrm{CC},\end{aligned} \tag{4.12}$$

where we used the identities (cf. [22])

$${}^*A_{ab}B^{ab} = A_{ab}{}^*B^{ab}, \qquad {}^{**}A^{ab} = -A^{ab}, \qquad {}^*A^{ABA'B'} = \mathrm{i}A^{ABB'A'}$$

for any two bivectors A^{ab} and B^{ab}. Inserting (4.12) into (4.7) gives (4.11), as claimed.

If we wish, it is also possible to define a *complexified Witten lift* as

$$\xi^\mu = e_a{}^\mu\sigma^a{}_{AA'}\lambda^A\bar{\lambda}^{A'}, \qquad \check{\Xi}_{ab} = (\check{\xi}^{\mathbb{C}}_{\mathrm{W}})_{ab} := -4\sigma_{[a}{}^{AA'}\sigma_{b]}{}^{BB'}\bar{\lambda}_{B'}\nabla_{BA'}\lambda_A. \tag{4.13}$$

Then, the relevant superpotential is

$$U(\mathcal{L}, \xi^{\mathbb{C}}_{\mathrm{W}}) = W := -\frac{2\mathrm{i}}{\kappa}\bar{\lambda}_{A'}\nabla\lambda_A \wedge \theta^{AA'}, \tag{4.14}$$

which is the (complex) Nester–Witten 2-form [19,23]. From the viewpoint of physical applications (proof of positivity of the Bondi or ADM mass, quasi-local definitions of momentum and angular momentum in general relativity, etc.), it is immaterial whether one

[2] With the exception of formula (4.13) below, we shall suppress hereafter the Infeld–van der Waerden symbols and adopt the standard identification $a = AA'$, $b = BB'$, etc., as is customary in the current literature (cf. [22]).

uses (4.14) or its real part (4.11), as its imaginary part turns out to be $-1/\kappa\, \mathrm{d_H}(\lambda_A\bar{\lambda}_{A'}\theta^a)$, which vanishes upon integration over a closed two-surface. Note, though, that (4.14) appears to relate more directly to Penrose quasi-local 4-momentum, when suitable identifications are made (cf. [23, p. 432]).

Remark 4.1. Note also that — modulo an inessential numerical factor — the Kosmann lift is (the real part of) the *dual* of the (complex) Witten lift, in the sense that

$$(\check{\xi}_{\mathrm{K}})_{ab} = -\tfrac{1}{2}\mathrm{Re}[{}^*(\check{\xi}^{\mathbb{C}}_{\mathrm{W}})_{ab}],$$

as can be easily checked on starting from Eq. (4.8) and the second equation of Eq. (4.13), whenever, of course, $\xi^a = \lambda^A\bar{\lambda}^{A'}$.

Remark 4.2. The theory developed herein is obviously tailored to the coupling with spinor fields described by the Dirac Lagrangian,

$$\mathcal{L}_{\mathrm{D}} := \left[\tfrac{1}{2}\mathrm{i}(\tilde{\Psi}\gamma^a\nabla_a\Psi - \widetilde{\nabla_a\Psi}\gamma^a\Psi) - m\tilde{\Psi}\Psi\right]\sqrt{g}\,\mathrm{d}s.$$

In the *purely metric* case, the total superpotential turns out to be

$$U(\mathcal{L}+\mathcal{L}_{\mathrm{D}}, \Xi) = U(\mathcal{L}, \Xi) + U(\mathcal{L}_{\mathrm{D}}, \Xi),$$

where

$$\begin{aligned} U(\mathcal{L}_{\mathrm{D}}, \Xi) &:= \frac{\mathrm{i}}{8}\tilde{\Psi}[(\gamma_a\gamma_b\gamma_c + 2\eta_{ac}\gamma_b)\xi^c]\Psi\,\Sigma^{ab} \\ &\equiv \frac{\mathrm{i}\sqrt{2}}{4}\xi_A^{A'}(\bar{\varphi}_{A'}\varphi_B - \bar{\psi}_B\psi_{A'})\Sigma^{AB} + \mathrm{CC}. \end{aligned}$$

The reader is referred to [3,9] for further details and notation.

Conversely, in the present *metric-affine* context, it can be readily shown that although the Dirac Lagrangian *does* enter the equations of motion (notably, the "second" *Einstein–Cartan* equation), it *does not* contribute to the total superpotential. From this fact one might mistakenly conclude that the Dirac fields do not contribute to the total conserved quantities. This conclusion is wrong because, although the Dirac Lagrangian does not contribute directly to the superpotential, in order to obtain the corresponding conserved quantities, one needs to integrate the superpotential on a solution, which in turn depends on the Dirac Lagrangian via its energy–momentum tensor and the second Einstein–Cartan equation.

5. First-order gravity

In the case of *vanishing torsion* ($T^a \equiv 0 \Leftrightarrow \nabla\Sigma^{ab} \equiv 0$), it is easy to see that Lagrangian (4.1) can be split into a total divergence plus a first-order covariant Lagrangian. In many contexts, the superpotential associated to this Lagrangian proved to give more physically reasonable answers than the Hilbert superpotential (cf. [9]).

For this reason and the sake of completeness, we now give the derivation of the aforementioned superpotential in the new geometrical framework outlined in Section 2.

The first-order covariant Lagrangian in question is (cf. [4,5])

$$\hat{\mathcal{L}} := -\frac{1}{2\kappa}(\hat{\Omega}_{ab} - Q_{ac} \wedge Q^c{}_b) \wedge \Sigma^{ab} \equiv \mathcal{L} + \frac{1}{2\kappa}\,\mathrm{d_H}(Q_{ab} \wedge \Sigma^{ab}), \tag{5.1}$$

where $\mathcal{L}$ is given by (4.1), $\hat{\Omega}_{ab} := \mathrm{d_H}\hat{\omega}_{ab} + \hat{\omega}_{ac} \wedge \hat{\omega}^c{}_b$ and $Q_{ab} := \omega_{ab} - \hat{\omega}_{ab}$, $\hat{\omega}_{ab}$ being a "background" (non-dynamical) spin connection. The corresponding Poincaré–Cartan form is

$$\begin{aligned}\Theta(\hat{\mathcal{L}}) &= \hat{\mathcal{L}} - \frac{1}{2\kappa}(\mathrm{d_V}\hat{\omega}_{ab} \wedge \Sigma^{ab} - \mathrm{d_V}\Sigma^{ab} \wedge Q_{ab})\\ &\equiv \Theta(\mathcal{L}) + \frac{1}{2\kappa}[\mathrm{d_H}(Q_{ab} \wedge \Sigma^{ab}) + \mathrm{d_V}Q_{ab} \wedge \Sigma^{ab} + \mathrm{d_V}\Sigma^{ab} \wedge Q_{ab}].\end{aligned}$$

Hence, the Noether current associated with a projectable vector field Ξ is

$$\begin{aligned}E(\hat{\mathcal{L}}, \Xi) &= E(\mathcal{L}, \Xi) + \frac{1}{2\kappa}[\pounds_\Xi Q_{ab} \wedge \Sigma^{ab} + \pounds_\Xi \Sigma^{ab} \wedge Q_{ab} - \xi \rfloor \mathrm{d_H}(Q_{ab} \wedge \Sigma^{ab})]\\ &= E(\mathcal{L}, \Xi) + \frac{1}{2\kappa}[\pounds_\Xi Q_{ab} \wedge \Sigma^{ab} + \pounds_\Xi (Q_{ab} \wedge \Sigma^{ab})\\ &\quad -\pounds_\Xi Q_{ab} \wedge \Sigma^{ab} - \xi \rfloor \mathrm{d_H}(Q_{ab} \wedge \Sigma^{ab})]\\ &= E(\mathcal{L}, \Xi) + \frac{1}{2\kappa}[\pounds_\Xi (Q_{ab} \wedge \Sigma^{ab}) - \xi \rfloor \mathrm{d_H}(Q_{ab} \wedge \Sigma^{ab})],\end{aligned} \tag{5.2}$$

ξ denoting, as usual, the projection of Ξ onto M. Now,

$$\pounds_\Xi (Q_{ab} \wedge \Sigma^{ab}) \equiv \pounds_\xi (Q_{ab} \wedge \Sigma^{ab}) = \xi \rfloor \mathrm{d_H}(Q_{ab} \wedge \Sigma^{ab}) + \mathrm{d_H}[\xi \rfloor (Q_{ab} \wedge \Sigma^{ab})]. \tag{5.3}$$

On substituting (5.3) into (5.2), we get

$$E(\hat{\mathcal{L}}, \Xi) = E(\mathcal{L}, \Xi) + \frac{1}{2\kappa}\mathrm{d_H}[\xi \rfloor (Q_{ab} \wedge \Sigma^{ab})],$$

whence

$$U(\hat{\mathcal{L}}, \Xi) := U(\mathcal{L}, \Xi) + \frac{1}{2\kappa}\xi \rfloor (Q_{ab} \wedge \Sigma^{ab}) \tag{5.4}$$

is recognized to be the superpotential associated with Lagrangian (5.1).

Remark 5.1. Note that, contrary to what happens in the purely natural context, no additional conditions need be imposed on the vector field Ξ here.

6. Discussion

This paper stresses the important role the theory of gauge-natural bundles plays in a significant issue of mathematical physics such as the definition of the gravitational energy and, more generally, of conserved quantities associated with the gravitational field, especially when coupled to spinor fields.

Besides providing a new gravitational superpotential in a gauge-natural context, the paper sheds some new light on the definition of the Nester–Witten 2-form and gives it an interpretation as a further, genuine gravitational superpotential.

Moreover, this paper shows that it is crucial in this context *not* to regard the metric as the fundamental gravitational field. Indeed, when considering the interaction between gravity and spinors one is forced to give up a purely natural formalism and instead consider a gauge-natural formalism in which one chooses the spin-tetrad (together with a spin connection, in a metric-affine formulation) as one's fundamental variable.

A parallel and analogous method of investigation is possible when dealing, in a gauge-natural context, with Legendre and dual Legendre transforms, for which the reader is referred to the recent and fundamental papers by Raiteri et al. [24] and Ferraris et al. [6].

Acknowledgements

P.M. acknowledges an EPSRC research studentship and a Faculty Research Studentship from the University of Southampton.

References

[1] M. Dubois-Violette, J. Madore, Conservation laws and integrability conditions for gravitational and Yang–Mills field equations, Comm. Math. Phys. 108 (1987) 213–223.

[2] L. Fatibene, M. Ferraris, M. Francaviglia, M. Godina, A geometric definition of Lie derivative for spinor fields, in: J. Janyška, I. Kolář, J. Slovák (Eds.), Proceedings of the Sixth International Conference on Differential Geometry and Its Applications, Brno, 1995, Masaryk University, Brno, 1996, pp. 549–558.

[3] L. Fatibene, M. Ferraris, M. Francaviglia, M. Godina, Gauge formalism for general relativity and fermionic matter, Gen. Rel. Grav. 30 (9) (1998) 1371–1389.

[4] M. Ferraris, M. Francaviglia, Covariant first-order Lagrangians, energy-density and superpotentials in general relativity, Gen. Rel. Grav. 22 (9) (1990) 965–985.

[5] M. Ferraris, M. Francaviglia, M. Mottini, Conserved quantities of the gravitational field in tetrad notation, Rend. Mat. (7) 14 (1994) 457–481.

[6] M. Ferraris, M. Francaviglia, M. Raiteri, Dual Lagrangian field theories, J. Math. Phys. 41 (4) (2000) 1889–1915.

[7] M. Ferraris, M. Francaviglia, O. Robutti, Energy and superpotentials in gravitational field theories, in: M. Modugno (Ed.), Atti del 6° Convegno Nazionale di Relatività Generale e Fisica della Gravitazione, Pitagora Editrice, Bologna, 1986, pp. 137–150.

[8] G. Giachetta, L. Mangiarotti, G. Sardanashvily, New Lagrangian and Hamiltonian Methods in Field Theory, World Scientific, Singapore, 1997.

[9] M. Godina, P. Matteucci, L. Fatibene, M. Francaviglia, Two-spinor formulation of first order gravity coupled to Dirac fields, Gen. Rel. Grav. 32 (1) (2000) 145–159.

[10] M.J. Gotay, J. Isenberg, J.E. Marsden, Momentum maps and classical relativistic fields. Part I. Covariant field theory, 1998. E-print: physics@xxx.lanl.gov (physics/9801019).

[11] J. Kijowski, A finite-dimensional canonical formalism in the classical field theory, Comm. Math. Phys. 30 (1973) 99–128.

[12] J. Kijowski, On a new variational principle in general relativity and the energy of the gravitational field, Gen. Rel. Grav. 9 (10) (1978) 857–877.

[13] J. Kijowski, W. Szczyrba, Multisymplectic manifolds and the geometrical construction of the Poisson brackets in the classical field theory, in: J.-M. Souriau (Ed.), Géométrie Symplectique et Physique Mathématique (Colloq. Internat. C.N.R.S., Aix-en-Provence, 1974), C.N.R.S, Paris, 1975, pp. 347–379.
[14] J. Kijowski, W. Szczyrba, A canonical structure for classical field theories, Comm. Math. Phys. 46 (1976) 183–206.
[15] J. Kijowski, W.M. Tulczyjew, A Symplectic Framework for Field Theories, Lecture Notes in Physics, Vol. 107, Springer, Berlin, 1979.
[16] I. Kolář, P.W. Michor, J. Slovák, Natural Operations in Differential Geometry, Springer, Berlin, 1993.
[17] A. Komar, Covariant conservation laws in general relativity, Phys. Rev. 113 (1959) 934–936.
[18] D. Krupka, Lagrange theory in fibered manifolds, Rep. Math. Phys. 2 (1971) 121–133.
[19] L.J. Mason, J. Frauendiener, The Sparling 3-form, Ashtekar variables and quasi-local mass, in: T.N. Bailey, R.J. Baston (Eds.), Twistors in Mathematics and Physics, Lecture Notes, Vol. 156, London Mathematical Society, Cambridge University Press, Cambridge, 1990, pp. 189–217.
[20] J.M. Nester, A new gravitational energy expression with a simple positivity proof, Phys. Lett. 83A (1981) 241–242.
[21] R. Penrose, Quasi-local mass and angular momentum in general relativity, Proc. Roy. Soc. London Ser. A 381 (1982) 53–62.
[22] R. Penrose, W. Rindler, Spinors and Space–Time, Vol. 1: Two-spinor Calculus and Relativistic Fields, Cambridge University Press, Cambridge, 1984.
[23] R. Penrose, W. Rindler, Spinors and Space–Time, Vol. 2: Spinor and Twistor Methods in Space–Time Geometry, Cambridge University Press, Cambridge, 1986.
[24] M. Raiteri, M. Ferraris, M. Francaviglia, General relativity as a gauge theory of orthogonal groups in three dimensions, in: P. Pronin, G. Sardanashvily (Eds.), Gravity, Particles and Space–Time, World Scientific, Singapore, 1996, pp. 81–98.
[25] A. Trautman, Noether equations and conservation laws, Comm. Math. Phys. 6 (1967) 248–261.
[26] R.M. Wald, General Relativity, The University of Chicago Press, Chicago, 1984.